Dust particle of mass $m = 2\times10^{-9}$ kg has electric potential energy

$$U(x) = E\left(10 - 5e^{-(x/a)^2}\right)$$

where

$$E = 6\times10^{-11}\ \text{J}$$

$$a = 3.5\times10^{-4}\ \text{m}$$

x = distance from tip of a wire

a) is $x=0$ a stable equilibrium?

$$F_x = -\frac{\partial U}{\partial x} = -\frac{2E}{a^2}\cdot x\cdot 5e^{-(x/a)^2}$$

if $x>0$, F_x is negative, pushes back to 0

$x<0$, F_x is positive, pushes back to 0

$\boxed{\text{yes, } x=0 \text{ is stable equilibrium}}$

b)

$$F_x = ma_x \rightarrow a_x = \frac{\partial^2 x}{\partial t^2} = -\frac{2E}{ma^2}\cdot x\cdot 5e^{-(x/a)^2}$$

We want to show

$$\frac{\partial^2 x}{\partial t^2} = -\overset{\text{const}}{(\ )}\,x$$

as condition for SHM

But we see $x\cdot e^{-(x/a)^2}$

Fortunately, if x is small, $x \ll a$, we can use an approximation. In general,

$$\text{if } z \ll 1,\ e^z \cong 1 + z + \frac{1}{2}z^2 + \dots$$

$\longrightarrow$

So, since $\frac{x}{a} << 1$,

$$e^{-\left(\frac{x}{a}\right)^2} \approx 1 - \left(\frac{x}{a}\right)^2 + \frac{1}{2}\left(\frac{x}{a}\right)^4 - \cdots$$

$$\approx 1 - \frac{x^2}{a^2}$$

$$\approx 1$$

In our case, that means the accel on particle in x-dir is approximately

$$\frac{\partial^2 x}{\partial t^2} \cong -\left(\frac{10E}{ma^2}\right) x \left[1 - \frac{x^2}{a^2} + \frac{1}{2}\frac{x^4}{a^4} - \cdots\right]$$

$$\cong -\left(\frac{10E}{ma^2}\right) x \left[1\right]$$

$$\frac{\partial^2 x}{\partial t^2} \cong -\left(\frac{10E}{ma^2}\right) x$$ therefore SHM

c) Frequency of motion given by

$$\frac{\partial^2 x}{\partial t^2} \cong -\omega^2 x$$

$$\rightarrow \omega = \sqrt{\frac{10E}{ma^2}} = \sqrt{\frac{10\left(6\times10^{-11}\ \text{kg}\cdot\frac{\text{m}^2}{\text{s}^2}\right)}{\left(2\times10^{-9}\ \text{kg}\right)\left(3.5\times10^{-4}\ \text{m}\right)^2}}$$

$$\boxed{\omega = 1.30\times10^{3}\ \frac{\text{rad}}{\text{s}}}$$