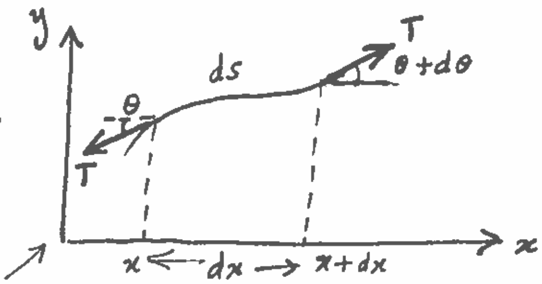
Lecture notes Fri March 08, 2019. (viv)

[5 min]

> Started by asking for examples of waves: ripples in a pond, vib. string, EM waves.

> Did standard "textbook" derivation of 1D eqn. for stretched string: [20 min]



<u>seek</u>: shape $y=y(x,t)$

Fig. shows "snapshot" for fixed time.

Let μ = mass per unit length of string assumed uniform.
Let T = tension throughout string presumed constant.

$\Sigma F_x = 0: \; T\cos(\theta+d\theta) \approx T\cos\theta$

$\Sigma F_y = ma_y: \; T\sin(\theta+d\theta) - T\sin\theta = m\dfrac{\partial^2 y}{\partial t^2}$

For small θ: $\sin\theta \approx \theta - \cancel{\dfrac{\theta^3}{3!}} + \cancel{\dfrac{\theta^5}{5!}} - \cdots$

$\tan\theta \approx \theta + \cancel{\dfrac{\theta^3}{3}} + \cancel{\dfrac{2\theta^5}{15}} + \cdots$

$\therefore$ Can replace $\sin\theta$ by $\tan\theta$.

$$T\left[\tan(\theta+d\theta) - \tan\theta\right] = \mu\, ds\, \frac{\partial^2 y}{\partial t^2}.$$

The geometric meaning of tangent is that it represent the instantaneous slope of the graph, $\dfrac{\partial y}{\partial x}$ at that point. Thus

$$T\left[\left.\frac{\partial y}{\partial x}\right|_{x+dx} - \left.\frac{\partial y}{\partial x}\right|_{x}\right] = \mu\, ds\, \frac{\partial^2 y}{\partial t^2}$$

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Take limit on both sides: $T \lim\limits_{dx\to 0}\left[\left.\dfrac{\partial y}{\partial x}\right|_{x+dx} - \left.\dfrac{\partial y}{\partial x}\right|_{x}\right] = \mu \dfrac{\partial^2 y}{\partial t^2} \lim\limits_{dx\to 0} ds$

Now, $\dfrac{\partial^2 y}{\partial x^2} = \lim\limits_{dx\to 0} \dfrac{\left.\frac{\partial y}{\partial x}\right|_{x+dx} - \left.\frac{\partial y}{\partial x}\right|_{x}}{dx}$; $ds = \sqrt{dx^2+dy^2}$, $dy = \dfrac{\partial y}{\partial x}dx + \dfrac{\partial y}{\partial t}\cancel{dt}^{\,0}$ since "snapshot"

$\therefore \lim\limits_{dx\to 0} ds = \lim\limits_{dx\to 0} dx\sqrt{1+\left(\dfrac{\partial y}{\partial x}\right)^2} = dx$ neglecting the second order terms and higher.

$$\therefore \; T\frac{\partial^2 y}{\partial x^2}\cancel{dx} = \mu\, \cancel{dx}\, \frac{\partial^2 y}{\partial t^2} \Rightarrow \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}}$$

I had them calculate the units of $\dfrac{\mu}{T}$, which is $\dfrac{m^2}{s^2}$. Let $V^2 = \dfrac{T}{\mu}$. Then $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{V^2}\dfrac{\partial^2 y}{\partial t^2}$!

V = propagation speed of wave, back and forth along string.

<u>3D</u>. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{V^2}\dfrac{\partial^2 u}{\partial t^2}$, $u = u(x,y,z,t)$

D'Alembert introduced $\Box = \nabla^2 - \dfrac{1}{V^2}\dfrac{\partial^2}{\partial t^2}$, so $\Box u = 0$ is the full-fledged wave eqn.

> In the remaining 25 min. I told them we don't need tensions or FBDs. Waves come about in much more general settings, such as E&M. I asked them to recall UP2 Maxwell's eqns.:

I upped the ante by telling them I'd never use poor Gauss in derivation below, and charged them to ponder on the rôle it may play in keeping the derivation rigorous

closed fluxes
$$\left\{\begin{aligned}\Phi_E^{\,c} &= \oint\!\!\!\oint_s \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}\\ \Phi_B^{\,c} &= \oint\!\!\!\oint_s \vec{B}\cdot d\vec{A} = 0\end{aligned}\right\}$$ Gauss' laws

vacuum

$$\oint \vec{E}\cdot d\vec{s} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{A}$$
green

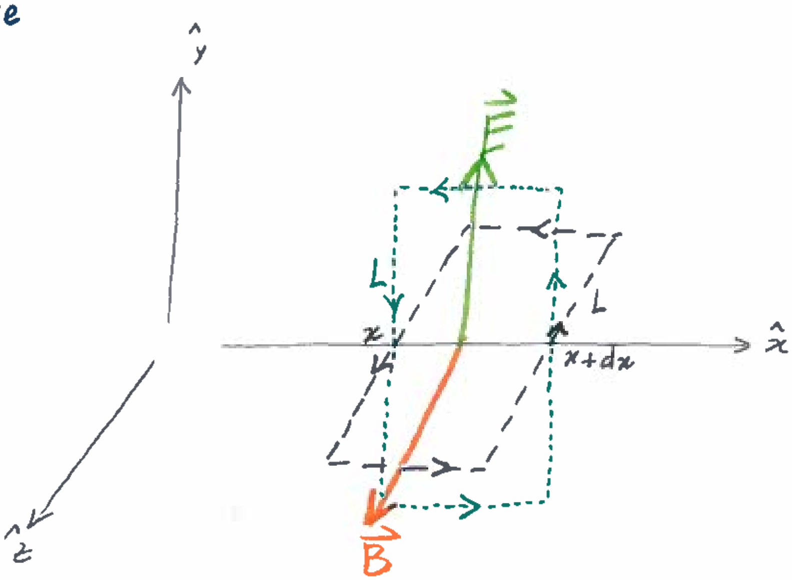
$$\oint \vec{B}\cdot d\vec{s} = \mu_0\epsilon_0\frac{\partial}{\partial t}\iint \vec{E}\cdot d\vec{A}$$
blue

Max-Amp $\oint_c \vec{B}\cdot d\vec{s} = \mu_0(i+i_D)$, $i_D = \epsilon_0\dfrac{\partial \Phi_E^{\,o}}{\partial t}$

$\Phi_E^{\,o} = \iint_s \vec{E}\cdot d\vec{A}$ open flux electric

Faraday $\oint_c \vec{E}\cdot d\vec{s} = -\dfrac{\partial}{\partial t}\Phi_B^{\,o}$,

$\Phi_B^{\,o} = \iint_s \vec{B}\cdot d\vec{A}$ open flux magnetic.



Assume: Plane polarized wave. (0 rad/m)

Note: RHR used to assign loop traverse directions.

$\Phi_E^{\,o} = L\,dx\,E\cos 0^\circ = E\,L\,dx$ $\boxed{2}$

$\Phi_B^{\,o} = L\,dx\,B\cos 0^\circ = B\,L\,dx$

Green: $E(x+dx)\cdot L\cdot\cos 0^\circ + 0 + E(x)\cdot L\cdot\cos 180^\circ + 0 = -\dfrac{\partial}{\partial t}(BL\,dx)$

cancel Ls, ÷ by dx: $\dfrac{E(x+dx)-E(x)}{dx} = -\dfrac{\partial B}{\partial t} \xrightarrow[dx\to 0]{\lim} \boxed{\dfrac{\partial E}{\partial x} = -\dfrac{\partial B}{\partial t}}$ — (*)

Blue: $B(x)\cdot L\cdot\cos 0^\circ + 0 + B(x+dx)\cdot L\cdot\cos 180^\circ + 0 = \mu_0\epsilon_0\dfrac{\partial}{\partial t}(EL\,dx)$

cancel Ls, ÷ by dx: $\dfrac{B(x)-B(x+dx)}{dx} = \mu_0\epsilon_0\dfrac{\partial E}{\partial t} \xrightarrow[dx\to 0]{\lim} \boxed{-\dfrac{\partial B}{\partial x} = \mu_0\epsilon_0\dfrac{\partial E}{\partial t}}$ — (†)

$\dfrac{\partial}{\partial x}$(*) and $\dfrac{\partial}{\partial t}$(†): $\dfrac{\partial^2 E}{\partial x^2} = -\dfrac{\partial^2 B}{\partial x\partial t}$ and $-\dfrac{\partial^2 B}{\partial t\partial x} = \mu_0\epsilon_0\dfrac{\partial^2 E}{\partial t^2}$.

I quoted Clairaut's theorem: for twice diff. functions mixed partial derivatives commute, $\dfrac{\partial^2 B}{\partial x\partial t} = \dfrac{\partial^2 B}{\partial t\partial x}$.

$\therefore\ \dfrac{\partial^2 E}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 E}{\partial t^2}$ where $c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}} = \dfrac{1}{\sqrt{\cancel{4\pi}\times 10^{-7}\,\frac{H}{m}\cdot\frac{1}{\cancel{4\pi}\times 9\times 10^{9}}\,\frac{F}{m}}} = 3\times 10^{8}\,\text{m/s}$ since $Hz = \dfrac{1}{\sqrt{LC}}$.

Homework: • Do $\dfrac{\partial}{\partial x}$(†) and $\dfrac{\partial}{\partial t}$(*) and get wave eqn. for B similarly!
• Ponder over the rôle played by Gauss' laws in this derivation.

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