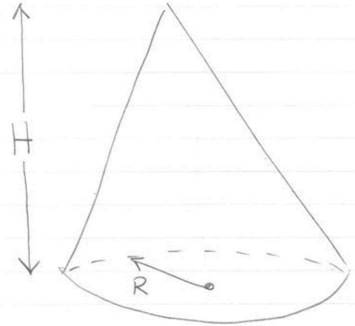

Cone of height H, radius R.

If it has uniform density ρ, where is center of mass?

- Must be along the central vertical axis of the cone, by symmetry. But how far above the base?

Break cone into slices. Consider a slice

h above ground
dh thickness

It has a radius

$$r = R\left(1 - \frac{h}{H}\right)$$

So this slice has

$$\text{volume} \quad dV = \pi r^2 dh = \pi R^2 \left(1 - \frac{h}{H}\right)^2 dh$$

$$\text{mass} \quad dm = \rho dV = \rho \pi R^2 \left(1 - \frac{h}{H}\right)^2 dh$$

Add up all slices to calculate mass of cone:

$$M = \int dm = \int_{h=0}^{h=H} \rho \pi R^2 \left(1 - \frac{h}{H}\right)^2 dh$$

$$= \rho \pi R^2 \int_{h=0}^{h=H} \left(1 - \frac{h}{H}\right)^2 dh$$

$$= \rho \pi R^2 \int_{h=0}^{h=H} \left(1 - 2\frac{h}{H} + \frac{h^2}{H^2}\right) dh$$

$$= \rho \pi R^2 \left[h - \frac{h^2}{H} + \frac{1}{3}\frac{h^3}{H^2} \right|_{h=0}^{h=H}$$

$$\underline{M = \tfrac{1}{3} \rho \pi R^2 H} \quad \text{total mass of cone.}$$

To calculate center of mass, we need to calculate

$$\text{height of center of mass} = \frac{\int h \, dm}{\text{total mass}}$$

$$= \frac{\int \rho \pi R^2 \left(1 - \frac{h}{H}\right)^2 h \, dh}{\frac{1}{3} \rho \pi R^2 H}$$

$$\text{height of center of mass} = \frac{\rho \pi R^2 \int_{h=0}^{h=H} \left(1 - \frac{h}{H}\right)^2 h \, dh}{\frac{1}{3}\rho \pi R^2 H}$$

$$= \frac{\rho \pi R^2 \int \left(h - 2\frac{h^2}{H} + \frac{h^3}{H^2}\right) dh}{\frac{1}{3}\rho \pi R^2 H}$$

$$= \frac{\rho \pi R^2 \left[\frac{1}{2}h^2 - \frac{2}{3}\frac{h^3}{H} + \frac{1}{4}\frac{h^4}{H^2} \right|_{h=0}^{h=H}}{\frac{1}{3}\rho \pi R^2 H}$$

$$= \frac{\rho \pi R^2 \left[\frac{1}{2}H^2 - \frac{2}{3}H^2 + \frac{1}{4}H^2 \right]}{\frac{1}{3}\rho \pi R^2 H}$$

$$= \frac{\frac{1}{12}\rho \pi R^2 H^2}{\frac{1}{3}\rho \pi R^2 H} = \boxed{\frac{1}{4}H}$$

center of mass of uniform cone.

Now, suppose the cone is denser at the top than the bottom, so that

$$\rho(h) = \rho_0 \left(1 + \frac{h}{H}\right)$$

In other words,

at bottom $\quad h = 0 \quad \rho = \rho_0$
at top $\quad h = H \quad \rho = 2\rho_0$

First, we need to calculate the mass of this cone.

$$M = \int dm = \int_{h=0}^{h=H} \rho_0 \left(1 + \frac{h}{H}\right) \pi R^2 \left(1 - \frac{h}{H}\right)^2 dh$$

$$= \rho_0 \pi R^2 \int_{h=0}^{h=H} \left(1 + \frac{h}{H}\right)\left(1 - \frac{2h}{H} + \frac{h^2}{H^2}\right) dh$$

$$= \rho_0 \pi R^2 \int_{h=0}^{h=H} \left(1 - \frac{h}{H} - \frac{h^2}{H^2} + \frac{h^3}{H^3}\right) dh$$

$$= \rho_0 \pi R^2 \left[h - \frac{1}{2}\frac{h^2}{H} - \frac{1}{3}\frac{h^3}{H^2} + \frac{1}{4}\frac{h^4}{H^3} \right|_{h=0}^{h=H}$$

$$= \rho_0 \pi R^2 \left[H - \frac{1}{2}H - \frac{1}{3}H + \frac{1}{4}H \right]$$

$$= \underline{\frac{5}{12}\rho_0 \pi R^2 H}$$

which <u>is</u> larger than mass of uniform cone.

Next, we need to calculate the integral

$$\int h\,dm = \int_{h=0}^{h=H} h\left(\pi R^2 \rho_0 \left[1+\frac{h}{H}\right]\left[1-\frac{h}{H}\right]^2\right) dh$$

$$= \rho_0 \pi R^2 \int h\left(1-\frac{h}{H}-\frac{h^2}{H^2}+\frac{h^3}{H^3}\right) dh$$

$$= \rho_0 \pi R^2 \int \left(h-\frac{h^2}{H}-\frac{h^3}{H^2}+\frac{h^4}{H^3}\right) dh$$

$$= \rho_0 \pi R^2 \left[\frac{1}{2}h^2-\frac{1}{3}\frac{h^3}{H}-\frac{1}{4}\frac{h^4}{H^2}+\frac{1}{5}\frac{h^5}{H^3}\right]_{h=0}^{h=H}$$

$$= \rho_0 \pi R^2 \left[\frac{1}{2}H^2-\frac{1}{3}H^2-\frac{1}{4}H^2+\frac{1}{5}H^2\right]$$

$$= \rho_0 \pi R^2 \left[\frac{7}{60}H^2\right]$$

So

$$\text{height of center of mass} = \frac{\int h\,dm}{\int dm} = \frac{\frac{7}{60}\rho_0 \pi R^2 H^2}{\frac{5}{12}\rho_0 \pi R^2 H}$$

$$= \boxed{\frac{7}{25}H}$$

vs. $\frac{1}{4}H = \frac{6}{24}H$ for a uniform cone.

So center of mass is a little higher, as it should be.