# Appendix B: Uncertainties, Error Analysis, and Significant Figures

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## 1 Need for Uncertainty Estimates

Physics, other sciences, and engineering are experimental disciplines. While theoretical calculations and predictions are made, ultimately they are checked by looking at measured values. **All measured values should include some estimate of the uncertainty**, so for example a length may be quoted as $(15.3 \pm 0.6)$ cm.

The following pages provide a brief tutorial on how to estimate uncertainties, and what to do with them. More complete information is online at `http://www.rit.edu/cos/uphysics/uncertainties/Uncertaintiespart1.html`. Full treatments are reserved for statistics courses.

## 2 Making Comparisons—The Normal Distribution

### 2.1 Variations in measurements

Suppose that you make several measurements of a physical quantity such as the period of a pendulum. You are not likely to get the same answer each time. Two main reasons for this are

1. The quantity being measured is somewhat variable. (E.g your height will vary depending on whether you stand straight and it is different when you get out of bed than when you have been standing all day; your mass varies as you breath in and out.)
2. The methods used for the measurement introduce some variation. (E.g. in measuring the pendulum period you rely on your visual observation to decide when to start and when to stop the watch.)

The measurement of the variations leads to the field of statistics. Typically we will take many measurements of a quantity and use the average as a best estimate.

$$x_{ave} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{1}$$

We also want an idea of how far the measurements were from the average. The terms *error*, *standard deviation*, and *uncertainty* are all used to describe the variation, and often are synonymous.

## 2.2 Random versus Systematic error

Suppose I measure the length of a wood block using a metal ruler in the following locations: Bogota, Mexico City, Nashville, Rochester, and Nome. I might get results that suggest that the wood block is longer when the country speaks English than when it speaks Spanish. In fact there is a *systematic error* relating to the temperature and the change in the length of the ruler in the different locations.

Systematic errors occur when an uncontrolled (unmeasured) variable affects the data so that the values are always too large or too small. In our example with the steel ruler, in hot countries the ruler expands, and so the measurement of a block will systematically be smaller than if the ruler were cold.

A famous example is the 1948 US presidential election. A newspaper conducted a telephone poll of voters on the day of the election and declared that Dewey had beaten Truman in the 1948 presidential election. In fact Truman won. Can you figure out what the systematic error was? We will not discuss systematic errors any further, because each experiment must be analyzed in detail to identify any systematic errors.

Instead we will discuss *random errors* since the field of statistics developed to treat the effects of random errors on our decisions. We will consider only the simplest (and most common) case of a Gaussian distribution of errors (also called a normal distribution, or a bell-shaped curve.) Measurements are centered on an average value with equal numbers above and below. If we plot a histogram of our measurements we get the well known *Gaussian curve*, also called a *normal curve*, or a *bell-shaped curve*.

Suppose we have two blocks, and repeat the measurement of the length of each block many times. We compute the same average length for both blocks and get a value of 4.45 cm. Figure 1 shows histograms of the two sets of lengths. The histograms are shown as Gaussian distributions. Underneath each graph is a shaded chart that suggests the graph. It is dark where the graph is large and fades to white as the graph drops to zero.

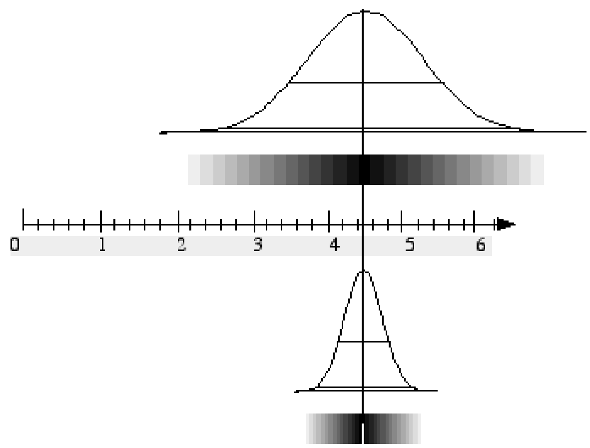


Figure 1: Measurements of the length of two blocks are made. Each block is measured many times measured many time and a histogram of the results is made. The average value is 4.45 cm in both cases, but the range of values is different. The shaded bar is another way to represent the average (center dark spot) and range of measurements.

## 2.3 Precision and Accuracy

Accuracy deals with how well the center of the curve matches the real value of what we are measuring. *Accurate* measurements have no systematic error. Suppose that the length of a block is supposed to be 4.48 cm. Figure 1 shows two measurements of the block, and in both cases the Gaussian curves are centered on 4.45 cm, close to the expected value, meaning that the measurements are quite accurate.

The top curve represents measurements with a wider spread of values than the bottom curve. We say that the bottom curve is a more *precise* measurement of the thickness. Precise measurements have a small spread of values, $\sigma$.

## 2.4 Standard Deviation Qualitatively

The Gaussian is asymptotic to the axis (infinitely wide). We need a method to specify the relative widths of the curves. The standard deviation, $\sigma$, is a measure of the width of the curves. The horizontal line in Figure 1 shows the standard deviation (from the center to where the line crosses the curve.) The line on the figure represents $x_{avg} \pm \sigma$.

We can represent the curve by a shaded box. It is darkest in the middle where most measurements occur, and fades out to zero as we go away from the center.

For Gaussians, approximately 2/3 of the measurements, 67%, lie within 1 standard deviation ($1\sigma$) of the center, $x_{avg} \pm \sigma$. If we go two standard deviations out($2\sigma$, $x_{avg} \pm 2\sigma$), 95% of the measurements are accounted for.[1]

## 2.5 Comparing Measurements.

Suppose we have four measurements:

A ($3.8 \pm 0.5$) cm, B ($5.1 \pm 0.5$) cm, C ($3.6 \pm 0.2$) cm, D ($4.7 \pm 0.3$) cm.

These are shown in Figure 2.

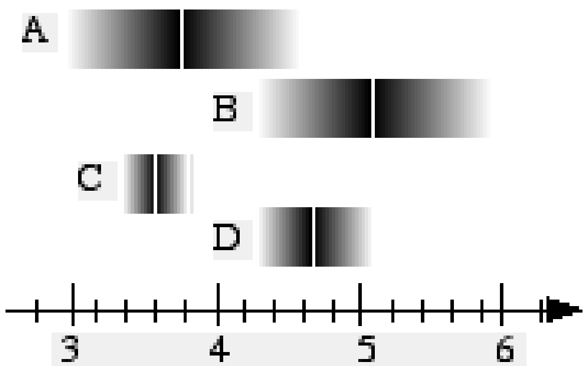


Figure 2: Four measurements of a length. The center white line in each bar shows the average, the bar shaded from black to gray shows the histogram of values and suggests the standard deviation.

If we don't have the standard deviations we can only say that measurements A and C are pretty close and the others seem different.

But with standard deviations we can make much better statements.

- A and C agree within $1\sigma$: A is between 3.3 cm and 4.3 cm while C is between 3.4 cm and 3.8 cm. The two ranges overlap. If they agree at the $1\sigma$ level they also agree at the $2\sigma$ level.
- A and B disagree within $1\sigma$: A is between 3.3 cm and 4.3 cm while B is between 4.6 cm and 5.6 cm. The two ranges do not overlap. However A and B agree at the $2\sigma$ level since A is between 2.8 and 4.8 cm while B is between 4.1 and 6.1 cm, and these ranges overlap.
- B and C disagree at the $2\sigma$ level: B is between 4.1 and 6.1 cm while C is between 3.2 cm and 4.0 cm. The two ranges do not overlap.

[1]In statistics courses this will be described as a level of confidence.

By following this procedure you should be able to say that at the $2\sigma$ level, $A = B, A = C, A = D, B = D$ but $B \neq C, C \neq D$.

# 3 Determining Uncertainty

Now that we have discussed what to do with the uncertainties, we need to discuss how to determine them. We will describe three methods, repeating measurements in order to get average deviation (or standard deviation), instrument limits of error, and estimations of uncertainty. Eventually we will use $\Delta x$ to represent any one of the uncertainties in $x$.

## 3.1 Repeating measurements

Suppose we want to measure the time for a piece of crumpled paper to fall to the floor from shoulder height. We could repeat the measurement several times and record the result as shown in Table 1.

Table 1: Time for a crumpled piece of paper to fall 2 meters

| | $t_i$ (sec) | $\|t_i - t_{avg}\|$(sec) |
|---|---|---|
| | 0.82 | 0.04125 |
| | 0.85 | 0.01125 |
| | 0.85 | 0.01125 |
| | 0.92 | 0.05875 |
| | 0.87 | 0.00875 |
| | 0.84 | 0.02125 |
| | 0.86 | 0.00125 |
| | 0.88 | 0.01875 |
| Average | 0.86125 | 0.0215625 |

There are $N = 8$ measurements and we can find the average, $x_{ave} = \frac{1}{N}\sum x_i$, to be 0.86125 sec.

For a simple measure of the range of values we will compute the **absolute value** of the deviation of each measurement from the average, then average these to get an *average deviation* of 0.0215625. In statistics a slightly more cumbersome computation is done to find the *standard deviation*. For our purposes average deviation is sufficient.

Recall the fuzzy bars in Figure 1. This suggests that there is no need to have so many digits in our answers. We will round our average deviation to one significant figure[2], 0.02, and

[2] Some instructors may require two significant figures.

round the average to match, 0.86, and then record the time as $(0.86 \pm 0.02)$ sec (average deviation).

Many calculators have statistical calculations included, and you can use these to get standard deviation. If you use EXCEL there are functions AVERAGE and STDEV that return the average and standard deviation.

You should clearly state which uncertainty calculation you are using when you report a result. If we used standard deviation for the data in Table 1 we would report the time as $(0.86 \pm 0.03)$ sec (using standard deviation.)

### 3.2 Instrument Limitation of Error

If you used a ruler to measure the width of a standard piece of paper, and repeated the measurement many times you would end up with a set of identical numbers, 21.6 cm. The average deviation would be zero, however that is not the value that you should give.

The smallest unit on your ruler, called the *least count*, is 0.1 cm. You may only be able to read to the nearest least count, or you may feel that you can estimate to some fraction of the least count, maybe half the least count. The smallest value that you are confident that you read is called the *instrument limit of error, or ILE*, and that is your uncertainty.

You and a friend may disagree on the ILE since that is a judgement. You might give the width of the paper as $(21.60 \pm 0.05)$ cm while your friend might give the width as $(21.6 \pm 0.1)$ cm. Both these answers would be fine.

### 3.3 Estimated error

You are given a stopwatch and told to time a 100 meter dash. The reading on the watch is 9.398 sec. What should you report for the time including uncertainty?

You cannot repeat any measurements, after all this was the gold medal race! The stopwatch has an ILE of 0.001 s. But that is not the uncertainty, since you know that you have some issues with using the watch.

- You started the watch when you heard the gun, and stopped it when you saw the winner cross the line. You know you have a reaction time. You might design a procedure to determine your reaction time, but you can't do that before you are asked for the time.
- You stopped the watch when the winner crossed the finish line. There is a lot of judgement required here in deciding which part of the body to concentrate on as it crosses the finish.

- You know that there is a systematic error involved since sound takes 0.294 s to travel 100 m. This means that you did not start the watch when the runners heard the gun, but shortly after. You can fix this by adding the sound travel time to the watch reading to get 9.692 s.

To set an uncertainty you must estimate the error that might arise from the first two items above, perhaps saying that by hand you expect the uncertainty to be 0.1 s. Then you can report the time as $(9.7 \pm 0.1)$ s.

If you watch a 100 meter dash you will notice that behind each racer is a speaker so that the systematic error is reduced to the time for sound to travel about 1 m, or 0.003 s. A light beam and photosensor replace a physical tape to indicate the end of the race. The electronics has a very short reaction time that may be negelected. Results are recorded to the hundreth of a second suggesting an estimated error of 0.01 sec.

### 3.4 Conflicts Between the Three Methods

Sometimes, as in the case of timing a race, different methods of getting the uncertainty result in different values. In general be pessimistic and take the largest of the three. Thus if you measure a time of 12.692 s using a watch with an ILE of 0.001 s, estimate your reaction time contribution as 0.05 s, and find an average deviation in results to be 0.4 s, report the result as $(12.7 \pm 0.4)$ s.

### 3.5 Form of the Answer

Report the answer so it is easy for a reader to understand.

1. Round the uncertainty to one significant digit.
2. Round the answer to agree with the uncertainty, i.e. agree to the same decimal place
3. Use the same units for the answer and the uncertainty.
4. Place the answer and uncertainty inside parentheses and the unit outside, as in $(12.7 \pm 0.4)$ s.
5. If the answer needs scientific notation, put the power of 10 outside the parentheses, such as $(1.62 \pm 0.03) \times 10^{-19}$ C.

# 4 Propagation of Error

After making measurements and deciding on the uncertainties you may have several quantities that we will call $x_{avg} \pm \Delta x$, $y_{avg} \pm \Delta y$, $z_{avg} \pm \Delta z$, etc. The uncertainties are labeled $\Delta x$ rather than $\sigma$ to indicate that they may be an ILE, an estimated uncertainty, or an average uncertainty and not necessarily a standard deviation.

From these measured values, and some constants $C_1, C_2$, etc. we compute a quantity $Q$, and want to estimate the uncertainty $\Delta Q$. The procedure outlined below will suffice for this course, there are much more sophisticated statistical procedures that may be required in upper level courses.

Use the average values to get the average value of $Q$. To get $\Delta Q$ use values for $x, y, z, \cdots$ that make $Q$ as large as possible, then subtract the average value.

Two new terms are used below: *absolute uncertainty* is just $\Delta x$ while *fractional uncertainty* is $\Delta x / x_{avg}$.

## 4.1 Addition or subtraction $Q = x + y - z + C_1$

Add absolute uncertainties—note that adding a constant does not affect the absolute uncertainty.

$$\Delta Q = \Delta x + \Delta y + \Delta z \tag{2}$$

## 4.2 Multiplication or Division $Q = C_1 xy/z$

Add fractional uncertainties—note that a constant has no fractional uncertainty.

$$\frac{\Delta Q}{Q} = \frac{\Delta x}{x_{avg}} + \frac{\Delta y}{y_{avg}} + \frac{\Delta z}{z_{avg}} \tag{3}$$

Frequently you then solve for the absolute uncertainty,

$$\Delta Q = Q \left( \frac{\Delta x}{x_{avg}} + \frac{\Delta y}{y_{avg}} + \frac{\Delta z}{z_{avg}} \right) \tag{4}$$

### 4.3 Powers $Q = C_1 x^m y^n$

Multiply fractional uncertainties by the absolute value of the power.

$$\frac{\Delta Q}{Q} = |m|\frac{\Delta x}{x_{avg}} + |n|\frac{\Delta y}{y_{avg}} \tag{5}$$

### 4.4 Other Functions

Follow the general rule—pick values to make $Q$ as large as possible then subtract the average value of $Q$.

e.g. $Q = \dfrac{5\cos\theta}{0.6 + \cos\theta}$ with $\theta = 35 \pm 5°$.

First get the average value of $Q$ by using 35°, $Q = 2.88606$.

To increase $Q$ we use $(35 - 5) = 30°$ with the cosine in the numerator and $(35 + 5) = 40°$ with the cosine in the denominator to get the largest $Q = 3.16983$. The difference is $0.28377 \simeq 0.3$ so we can say $Q = (2.9 \pm 0.3)$.

### 4.5 Common Sense

**You should record an uncertainty in every measured value**. However in many experiments some of the measured quantities are much more uncertain than others, so in the propagation of error focus of the main contributors.

**e.g. 1** $Q = x + y$ with $x = (12.2 \pm 0.5)$ cm and $y = (7.124 \pm 0.002)$ cm. Ignore the uncertainty in $y$

$Q = (19.3 \pm 0.5)$ cm

**e.g. 2** $V = \pi r^2 h$ with $r = (0.153 \pm 0.008)$ cm and $h = (146 \pm 0.5)$ cm

For multiplication we need fractional uncertainties, $\Delta r/r = 0.0523, \Delta h/h = 0.0034$. Neglect the very small fractional uncertainty in height.

$V = 10.737$ cm$^3$ and to get the uncertainty in volume using Equation 4, $\Delta V = 10.737(0.0523) = 0.562 \approx 0.6$ cm$^3$.

Hence $V = (10.7 \pm 0.6)$ cm$^3$.

**e.g. 3** $Q = x + C\sin\theta$ with $x = (12 \pm 2)$ cm, $C = 0.3$ cm, and $\theta = 65 \pm 30°$

The second term is going to be less than one, and with the values given the error in the first term will dominate. Hence $Q = (12 \pm 2)$ cm and in this case the angle does not matter at all.

## 5 Significant Figures

The rules for propagation of errors hold true for cases when we are in the lab, but doing propagation of errors is time consuming. The rules for significant figures allow a much quicker method to get results that are approximately correct even when we have no uncertainty values. They are typically what you use in doing textbook problems.

A significant figure is any digit 1 to 9 and any zero which is not a place holder. Thus, in 1.350 there are 4 significant figures since the zero is not needed to make sense of the number. In a number like 0.00320 there are 3 significant figures—the first three zeros are just place holders. However the number 1350 is ambiguous. You cannot tell if there are 3 significant figures—the 0 is only used to hold the units place—or if there are 4 significant figures and the zero in the units place was actually measured to be zero.

How do we resolve ambiguities that arise with zeros when we need to use zero as a place holder as well as a significant figure? Suppose we measure a length to three significant figures as 8000 cm. Written this way we cannot tell if there are 1, 2, 3, or 4 significant figures. To make the number of significant figures apparent we use scientific notation, $8 \times 10^3$ cm (which has one significant figure), or $8.00 \times 10^3$ cm (which has three significant figures), or whatever is correct under the circumstances.

We start then with numbers each with their own number of significant figures and compute a new quantity. How many significant figures should be in the final answer? In doing running computations we maintain numbers to many figures, but we must report the answer only to the proper number of significant figures.

In the case of addition and subtraction we can best explain with an example. Suppose one object is measured to have a mass of 9.9 gm and a second object is measured on a different balance to have a mass of 0.3163 gm. What is the total mass?

We write the numbers with question marks at places where we lack information. Thus `9.9????` gm and `0.3163?` gm. Adding them with the decimal points lined up we see

```
  9.9????
  0.3163?
 --------
 10.2????
```

So we write the sum as 10.2 gm.

In the case of multiplication or division we can use the same idea of unknown digits. Thus the product of 3.413? cm and 2.3? cm can be written in long hand as

```
 3.413?
 2.3?
--------
 ..?????
 10239?
 6826?
--------
 78????
```

So the result is 7.8 $cm^2$.

We need a short rule for multiplication and division. The short rule is that the answer will contain a number of significant figures equal to the number of significant figures in the entering number having the least number of significant figures. In the above example 2.3 had 2 significant figures while 3.413 had 4, so the answer is given to 2 significant figures, 7.8 $cm^2$.

It is important to keep these concepts in mind as you use calculators with 8 or 10 digit displays if you are to avoid mistakes in your answers and to avoid the wrath of physics instructors everywhere. A good procedure to use is to use use all digits (significant or not) throughout calculations, and only round off the answers to appropriate “sig fig.”

# 6 Glossary of Terms

| Term | Brief Definition |
|---|---|
| Absolute error | The actual error in a quantity, having the same units as the quantity. Thus if $c = (2.95 \pm 0.07)$ m/s, the absolute error is 0.07 m/s. See Relative Error. |
| Accuracy | How close a measurement is to being correct. For gravitational acceleration near the earth, g = 9.7 m/s$^2$ is more accurate than g = 9.532706 m/s$^2$. See Precision. |
| Average | When several measurements of a quantity are made, the sum of the measurements divided by the number of measurements. |
| Average Deviation | The average of the absolute value of the differences between each measurement and the average. See Standard Deviation. |
| Deviation | A measure of range of measurements from the average. Also called error or uncertainty. |
| Error | A measure of range of measurements from the average. Also called deviation or uncertainty. |
| Estimated Uncertainty | An uncertainty estimated by the observer based on his or her knowledge of the experiment and the equipment. This is in contrast to ILE, standard deviation or average deviation. |
| Fractional uncertainty | The ratio of absolute error to the average, $\Delta x/x$. This may also be called percentage error or relative uncertainty. If $c = (2.95 \pm 0.07)$ m/s, the fractional uncertainty is $0.07/2.95 = 0.024$. See Absolute Error. |
| Gaussian Distribution | The familiar bell-shaped distribution. Simple statistics assumes that random errors are distributed in this distribution. Also called Normal Distribution. |
| Instrument Limit of Error (ILE) | The smallest reading that an observer can make from an instrument. This is generally smaller than the Least Count. |
| Least Count | The size of the smallest division on a scale. Typically the ILE equals the least count or 1/2 or 1/5 of the least count. |
| Normal Distribution | The familiar bell-shaped distribution. Simple statistics assumes that random errors are distributed in this distribution. Also called Gaussian Distribution. |
| Percentage Error | The ratio of absolute error to the average, $\Delta x/x$ expressed as a %. This may also be called fractional uncertainty or relative uncertainty. If $c = (2.95 \pm 0.07)$ m/s, the percentage error is $(0.07/2.95)100\% = 2.4\%$. See Absolute Error. |

| Term | Brief Definition |
|---|---|
| Precision | The number of significant figures in a measurement. For gravitational acceleration near the earth, g = 9.532706 m/s$^2$ is more precise than g = 9.7 m/s$^2$. Greater precision does not mean greater accuracy! See Accuracy. |
| Propagation of Errors | Given independent variables each with an uncertainty, the method of determining an uncertainty in a function of these variables. |
| Random Error | Deviations from the "true value" can be equally likely to be higher or lower than the true value. See Systematic Error. |
| Range of Possible True Values | Measurements give an average value, $x_{avg}$ and an uncertainty, $\Delta x$. At the $2\sigma$ level the range of possible true values is from $x_{avg} - 2\Delta x$ to $x_{avg} + 2\Delta x$. We will be correct using this method 95% of the time. |
| Relative Error | The ratio of absolute error to the average, $\Delta x/x$. This may also be called percentage error or fractional uncertainty. If $c = (2.95 \pm 0.07)$ m/s, the relative error is $0.07/2.95 = 0.024$. See Absolute Error. |
| Significant Figures | All non-zero digits plus zeros that do not just hold a place before or after a decimal point. |
| Standard Deviation | The statistical measure of uncertainty. See Average Deviation. Symbol is $\sigma$ |
| Systematic Error | A situation where all measurements fall above or below the "true value". Recognizing and correcting systematic errors is very difficult, and is specific to each experiment. See Random error. |
| Uncertainty | A measure of range of measurements from the average. Also called deviation or error. |