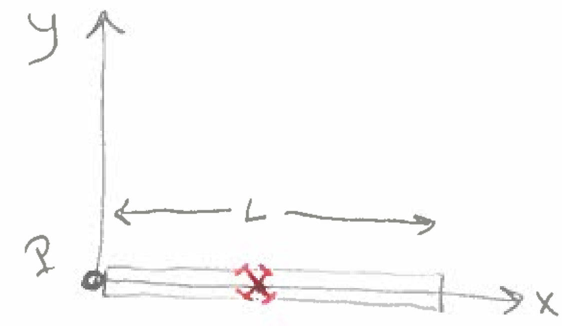


A uniform thin rod of length L and mass M sits attached to a pivot at the origin.

What is moment of inertia around the pivot?

$$I = \frac{1}{3} M L^2$$

Where is the center of mass of the rod? At the geometric center of the rod, marked with an ✕ in figure above.

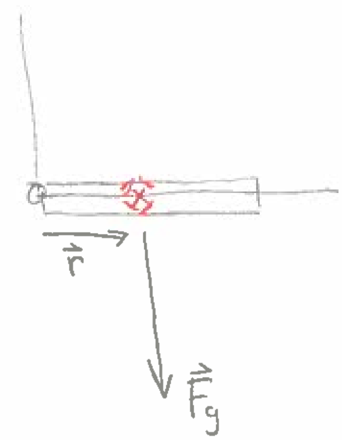
$$\text{Center of mass} = \frac{L}{2}\hat{\imath} + 0\hat{\jmath}$$

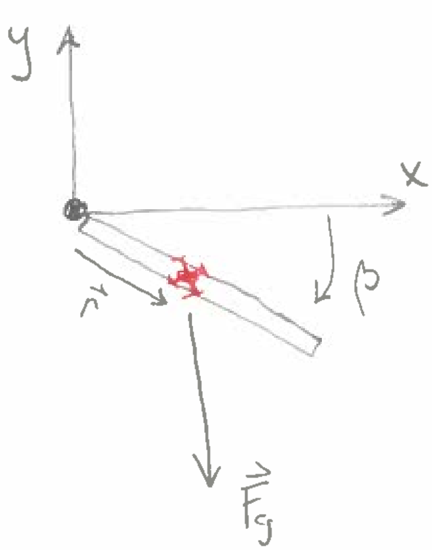
Gravity acts on the center of mass of the rod, so the torque around the pivot P is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

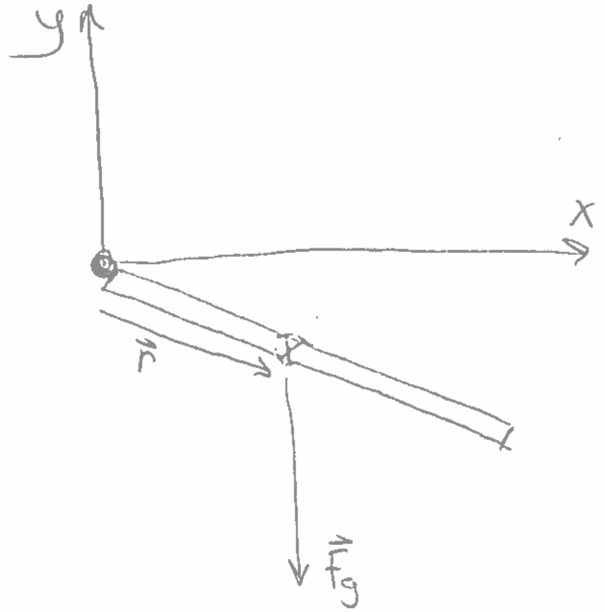
$$= \left|\frac{L}{2}\right| |mg| \sin 90^\circ \ \underline{\text{into}}$$

$$= \frac{L}{2} Mg \ \text{into page}$$

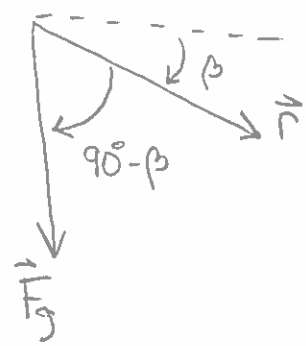




Now the rod swings down due to this torque. At one moment, the rod has swung by angle β.



The angle between $\vec{r}$ and $\vec{F}$ is $90° - \beta$



Position of center of mass is now

$$\text{CoM } x = r\cos\beta = \frac{L}{2}\cos\beta$$
$$y = -r\sin\beta = -\frac{L}{2}\sin\beta$$

The torque on the rod now is

$$\vec{\tau} = \vec{r} \times \vec{F} = |r||F|\sin(90-\beta) = |r||F|\cos\beta$$
$$= \left(\frac{L}{2}\right)(Mg)\cos\beta \ \underline{\text{into}} \text{ page}$$

If the rod swings from $\beta = 0°$ to $\beta = 40°$, the work done by the gravitational torque is

$$\text{Work} = \int \vec{\tau} \cdot d\vec{\theta} = \int \left(\frac{L}{2} Mg\cos\beta \ \underline{\text{into}}\right) \cdot \left(d\theta \ \underline{\text{into}}\right)$$

$$= \int_{\beta=0}^{\beta=40°} \frac{L}{2} Mg\cos\beta \, d\beta \cdot 1 \leftarrow \text{both vectors point in same direction}$$

$$= \frac{L}{2} Mg \int_{\beta=0}^{\beta=40°} \cos\beta \, d\beta$$

$$= \frac{L}{2} Mg \sin\beta \Big|_0^{40°} = \frac{L}{2} Mg \left(\sin 40° - \sin 0°\right)$$

The angular velocity of the rod is given by

$$\text{Work} = KE_f - KE_i$$

$$= \frac{1}{2} I \omega_f^2 - 0$$

$$\rightarrow \frac{1}{2} I \omega_f^2 = \text{Work} = \frac{L}{2} Mg \sin 40^\circ$$

$$\rightarrow \omega_f^2 = \frac{\frac{L}{2} Mg \sin 40^\circ}{\frac{1}{2} I} = \frac{\frac{L}{2} Mg \sin 40^\circ}{\frac{1}{2} \cdot \frac{1}{3} ML^2}$$

$$\omega_f^2 = \frac{3g \sin 40^\circ}{L}$$

$$\rightarrow \boxed{\omega_f = \sqrt{\frac{3g \sin 40^\circ}{L}}}$$